GCSE Maths – Algebra

Solving Quadratic Equations

Notes

WORKSHEET

Solving Quadratic Equations

Quadratic equations

Quadratic equations are equations in the form

$$\boldsymbol{ax^2 + bx + c = 0}$$

where a, b and c are integers and $a \neq 0$. This equation can be solved by **factorisation,** by method of **completing the square** (higher only) and by using the **quadratic formula** (higher only).

Solving quadratic equations by factorising

For a reminder on how to factorise, see the revision notes for **Algebra – Factorising Linear and Quadratic Expressions.**

Example: Solve the quadratic equation $2x^2 - 8x = 0$

1. **Factorise** the **common factor** out.

   The common factors in both terms are 2 *and* x *so we factorise out* $2x$*:*

$$2x^2 - 8x = 0$$
$$2x(x - 4) = 0$$

2. **Solve** the equation by equating both factors to 0.

$$2x = 0 \quad \text{or} \quad x - 4 = 0$$
$$x = 0 \qquad\qquad x = 4$$

Hence, the possible solutions for x *are* $\boldsymbol{x = 0}$ and $\boldsymbol{x = 4}$.

Example: Solve the quadratic equation $x^2 + 23x = 0$

1. **Factorise** the **common factor** out.

   The common factor in both terms is x *so we factorise out* x*:*

$$x^2 + 23x = 0$$
$$x(x + 23) = 0$$

2. **Solve** the equation by equating both factors to 0.

$$x = 0 \quad \text{or} \quad x + 23 = 0$$
$$x = 23$$

Hence, the possible solutions for x *are* $\boldsymbol{x = 0}$ and $\boldsymbol{x = 23}$.

In the case where $a = 1, b \neq 0$ and $c \neq 0$, the equation

$$\boldsymbol{ax^2 + bx + c = 0}$$

will have the form

$$\boldsymbol{x^2 + bx + c = 0.}$$

This needs to be solved by **factorisation** in the form of $(x+p)(x+q) = 0$ where $p \times q = c$ and $p + q = b$.

Example: Solve the quadratic equation $x^2 + 7x + 12 = 0$

1. If the value of a in the equation is 1, we can start by writing down the brackets in the form of $(x+p)(x+q) = 0$. Leave out the value of p and q first since we will fill this out later. Ensure that **the x in the bracket**, when **multiplied with each other**, gives the **original quadratic** equation.

   For instance, in this example, our quadratic is x^2. Hence, we write our brackets as:

   $$x^2 + 7x + 12 = (x+p)(x+q) = 0$$

2. **List** all the possible **factor pairs** for c. Factor pair means a pair of integers which when multiplied together is equal to c.

   In this example, $c = 12$. The factor pairs for 12 are:

   $$1 \times 12$$
   $$2 \times 6$$
   $$3 \times 4$$

3. **Identify** factor pairs which sum together to equal b.

   In this example, $b = 7$.

   $$1 + 12 \neq 7$$
   $$2 + 6 \neq 7$$
   $$\boldsymbol{3 + 4 = 7}$$

   *So, the correct factor pair is **3** and **4**.*

4. **Substitute** the correct factor pair as p and q in the bracket form that was setup in step 1.

   $$x^2 + 7x + 12 = (x+p)(x+q) = 0$$
   $$(x+3)(x+4) = 0$$

5. **Solve** the equation by equating both factors to 0.

   $$x + 3 = 0 \qquad \text{or} \qquad x + 4 = 0$$
   $$x = -3 \qquad\qquad x = -4$$

   Hence, the possible solutions for x are $\boldsymbol{x = -3}$ and $\boldsymbol{x = -4}$.

If the coefficient a of x^2 in the general form $\boldsymbol{ax^2 + bx + c = 0}$ is a common factor of the other terms b and c then it can be factorised and divided out of the expression before solving the quadratic using the method for equations of the form $\boldsymbol{x^2 + bx + c = 0}$.

Example: Solve the quadratic equation $3x^2 + 30x + 48 = 0$

1. **Find** a common factor and **factorise** it out of the equation.

$$3x^2 + 30x + 48 = 0$$
$$3(x^2 + 10x + 16) = 0$$

Divide both sides of the expression by 3*:*

$$x^2 + 10x + 16 = 0$$

2. After taking out the common factor, it is important to note that we have **new values for** $\boldsymbol{a}, \boldsymbol{b}$ **and** $\boldsymbol{c}$. These values refer to the simplified equation inside the bracket.

For the equation $x^2 + 10x + 16 = 0$*, comparing to the general form* $ax^2 + bx + c = 0$*, we have* $a = 1,\ b = 10$ *and* $c = 16$*.*

3. Write the new simplified equation in the form of $\boldsymbol{(x + p)(x + q) = 0}$.

In this example, our quadratic is x^2*. Hence, we write our brackets as:*

$$x^2 + 10x + 16 = (x + p)(x + q) = 0$$

4. **List out** all the possible **factor pairs** for c.

In this example, $c = 16$*. The factor pairs for* 16 *are:*

$$1 \times 16$$
$$2 \times 8$$
$$4 \times 4$$

5. **Look** for factor pairs which have the same value as b when added together.

In this example, $b = 10$*:*

$$1 + 16 \neq 10$$
$$2 + 8 = 10$$
$$4 + 4 \neq 10$$

So, the correct factor pair is 2 *and* 8*.*

6. Substitute the correct factor pair as p and q in the bracket form from Step 3.

$$(x + p)(x + q) = (x + 2)(x + 8) = 0$$

7. **Solve** the equation by equating both factors to 0.

$$x + 2 = 0 \quad \text{or} \quad x + 8 = 0$$
$$x = -2 \qquad\qquad x = -8$$

Hence, the possible solutions are for x are $\boldsymbol{x = -2}$ and $\boldsymbol{x = -8}$.

If a common factor cannot be found and $a \neq 1$, we require a different method to factorise the quadratic equation before we can find the possible solutions.

Example: Solve the quadratic equation $3x^2 + 8x + 4 = 0$

1. When $a \neq 1$ and there is **no common factor** which can be factored out of the equation, first we need to ensure the equation is in the form of $ax^2 + bx + c = 0$. Then, list down the values of a, b and c.

$$3x^2 + 8x + 4 = 0$$

$$a = 3,\ b = 8 \text{ and } c = 4$$

2. **Multiply** the **value of a** and c.

$$3 \times 4 = 12$$

3. **List down** all the possible **factor pairs** of the multiplied value of a and c found in the previous step.

The factor pairs of 12 are: 1×12, 2×6, 3×4

4. Identify which factor pair, when **added together**, gives the **value of b**.

In this example, $b = 8$. The sum for the factor pairs of 12:

$$1 + 12 \neq 8$$
$$2 + 6 = 8$$
$$3 + 4 \neq 8$$

In this case, the correct factor pair would be 2 and 6 since they give the same value as b.

5. **Write the correct factor pair** (found in Step 4) as **coefficient of x**, replacing bx in the original equation.

Original equation: $3x^2 + 8x + 4 = 0$
Substitute $8x$ *with the correct factor pair*: $3x^2 + 6x + 2x + 4 = 0$

6. Find a common factor for the first 2 terms and the last 2 terms and **factorise them separately**. Ensure that the equations in the brackets are **similar** to each other.

The common factor for the first 2 terms is $3x$. The common factors in the last 2 terms is 2:

$$3x\,(x + 2) + 2\,(x + 2) = 0$$

7. Present the equation as the product of two brackets. The **first bracket** will be the brackets we have made in the **previous step**. The **second bracket** will be made up from the terms which are coefficients of the brackets in the previous step.

$$3x\,(x + 2) + 2\,(x + 2) = 0$$
$$\Rightarrow (x + 2)(3x + 2) = 0$$

8. Solve the values for x by equating the factors to 0.

$x + 2 = 0$ or $3x + 2 = 0$

$\boldsymbol{x = -2}$ $\qquad$ $\boldsymbol{x = -\frac{2}{3}}$

Now, let us look at another example to further understand this method.

Example: Solve the quadratic equation $6x^2 = 8 - 13x$

1. When $a \neq 1$ and there is **no common factor** which can be factored out of the equation, first we need to ensure the equation is in the form of $ax^2 + bx + c = 0$. Then, list down the values of a, b and c.

$$6x^2 + 13x - 8 = 0$$

$$a = 6,\ b = 13 \text{ and } c = -8$$

2. **Multiply the value of a and c.**

$$ac = 6 \times -8 = -48$$

3. **List down** all the possible **factor pairs** of the multiplied value of a and c found in the previous step.

The factor pairs of -48 are:

$$-1 \times 48 \quad or \quad -48 \times 1$$
$$-2 \times 24 \quad or \quad -24 \times 2$$
$$-3 \times 16 \quad or \quad -16 \times 3$$
$$-4 \times 12 \quad or \quad -12 \times 4$$
$$-6 \times 8 \quad or \quad -8 \times 6$$

4. Identify which factor pair which, when **added together**, gives the **value of b**.

In this example, $b = 13$.

The sum for the factor pairs of -48:

$$-1 + 48 \neq 13 \quad or \quad -48 + 1 \neq 13$$
$$-2 + 24 \neq 13 \quad or \quad -2 + 2 \neq 13$$
$$-3 + 16 = 13 \quad or \quad -16 + 3 \neq 13$$
$$-4 + 12 \neq 13 \quad or \quad -12 + 4 \neq 13$$
$$-6 + 8 \neq 13 \quad or \quad -8 + 6 \neq 13$$

In this case, the correct factor pair would be -3 and 16 since they give the same value for b.

5. **Write the correct factor pair** (found in Step 4) as **coefficient of x**, substituting bx in the original equation.

$Original\ equation:\ 6x^2 + 13x - 8$
$Substitute\ 13x\ with\ the\ correct\ factor\ pair: 6x^2 - 3x + 16x - 8 = 0$

6. Find a common factor for the first 2 terms and the last 2 terms and **factorise them separately**. Ensure that the equations in the brackets are **similar** to each other.

In this case, the common factor for the first two terms is $3x$ and the common factor for the last two terms is 8.

$$3x\,(2x - 1) + 8(2x - 1) = 0$$

Note that the brackets created here are similar to each other.

7. Present the equation as the product of two brackets. The **first bracket** will be the brackets we have made in the **previous step**. The **second bracket** will be made up from the terms which are coefficients of the brackets in the previous step.

$$3x\,(2x-1)+8(2x-1)=0$$
$$(2x-1)(3x+8)=0$$

8. Solve the values for x by equating the factors to 0.

$$2x-1=0 \qquad \text{or} \qquad 3x+8=0$$
$$x=\frac{1}{2} \qquad\qquad\qquad x=-\frac{8}{3}$$

Hence, the possible solutions for x are $\boldsymbol{x=\frac{1}{2}}$ and $\boldsymbol{x=-\frac{8}{3}}$.

Solving quadratic equations by completing the square (Higher only)

Completing the square method can be used when a quadratic equation **cannot be easily factorised**. It is often expressed in the form of $(x+p)^2+q$, where p and q can be any positive or negative numbers.

The normal form of a quadratic equation is $\boldsymbol{ax^2+bx+c}$. This equation can be **transformed** to the $(x+p)^2+q$ form using the following steps:

$$x^2+bx+c=(x+p)^2+q \qquad [1]$$
$$x^2+bx+c=x^2+2px+p^2+q \qquad [Expand\ the\ bracket\ on\ RHS]$$

Now, we can compare the value of b and c directly.
From the equation, if we compare the coefficient of x on each side we get:

$$b=2p$$
$$p=\frac{b}{2} \qquad [Rearrange\ the\ equation\ for\ p]$$

Now, let us find q by comparing the constant terms on each side of the equation. From the equation above, we know that:

$$c=p^2+q$$
$$q=-p^2+c \qquad [Rearrange\ the\ equation\ for\ q]$$
$$q=-\left(\frac{b}{2}\right)^2+c \qquad [Substitute\ the\ value\ of\ p]$$

Now, if we plug in p and q into equation [1], we get:

$$x^2+bx+c=\left(x+\frac{b}{2}\right)^2-\left(\frac{b}{2}\right)^2+c$$

This equation proves to us that we can **directly transform** the ax^2+bx term to the completing the square form by **plugging in the b value** into this formula: $\left(x+\frac{b}{2}\right)^2-\left(\frac{b}{2}\right)^2$.
This will be explained further in the example.

Example: By completing the square, express the quadratic equation $x^2 + 10x + 4 = -5$ in the form of $(x+p)^2 + q$. Then, find the exact value for x.

1. **Ensure** that the quadratic equation is in the form of $ax^2 + bx + c = 0$. If the quadratic equation is not in this form, we need to **rearrange** the equation. Then, determine the value of a, b and c.

$$x^2 + 10x + 4 = -5$$
$$x^2 + 10x + 9 = 0$$

$$a = 1, \qquad b = 10, \qquad c = 9$$

2. **Convert** the $ax^2 + bx$ form into the completing the square form by plugging the b value into **this formula:** $\left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$.

$$x^2 + 10x = \left(x + \frac{10}{2}\right)^2 - \left(\frac{10}{2}\right)^2$$

3. **Simplify** the expression.

$$\left(x + \frac{10}{2}\right)^2 - \left(\frac{10}{2}\right)^2 = (x+5)^2 - (5)^2 = (x+5)^2 - 25$$

4. Add the value of c at the end of the expression and **simplify** it.

$$x^2 + 10x + 9 = (x+5)^2 - 25 + 9 = (x+5)^2 - 16$$

Now the expression is in the form of $(x+p)^2 + q$ where $p = 5$ and $q = -16$.

5. **Solve** the equation by equating the expression to 0.

$$x^2 + 10x + 9 = 0$$
$$(x+5)^2 - 16 = 0$$

Rearrange the equation to isolate x on one side of the expression:

$$(x+5)^2 = 16$$
$$(x+5) = \sqrt{16}$$
$$x + 5 = \pm 4$$
$$x = \pm 4 - 5$$

$$x = 4 - 5 \qquad or \qquad x = -4 - 5$$
$$x = -1 \qquad\qquad\qquad x = -9$$

Hence, the possible solutions for x are $\boldsymbol{x = -1}$ and $\boldsymbol{x = -9}$.

In cases where the coefficient of a in a quadratic equation **is not equal to 1**, it is often easier to **factorise out** the coefficient of a from the term $ax^2 + bx$, shown in the example below. Take extra care if the value of b is **negative**. You should write the answers in **fraction** or **surd** form unless the question states otherwise. If the question does not ask for an exact answer, you can give the answer up to **3 significant figures**.

Example: By completing the square, express the following quadratic equation in the form of $(x+p)^2+q$. Then, find the exact value for x.

$$3x^2 - 12x - 15 = 0$$

1. **Factorise** the coefficient of a from the term $ax^2 + bx$.

$$3x^2 - 12x - 15 = 0$$
$$3(x^2 - 4x) - 15 = 0$$

$$a = 1, \quad b = -4, \quad c = -15$$

2. **Convert** the $ax^2 + bx$ term in the bracket into completing the square form.

$$x^2 - 4x = \left(x + \frac{-4}{2}\right)^2 - \left(\frac{-4}{2}\right)^2$$

3. **Simplify** the expression.

$$x^2 - 4x = \left(x + \frac{-4}{2}\right)^2 - \left(\frac{-4}{2}\right)^2 = (x-2)^2 - (-2)^2 = (x-2)^2 - 4$$

4. **Substitute** the expression back to the $ax^2 + bx$ form in the bracket (from step 1).

$$3(x^2 - 4x) - 15 = 0$$
$$3[(x-2)^2 - 4] - 15 = 0$$

5. **Simplify** the equation.

$$3(x-2)^2 - 12 - 15 = 0$$
$$3(x-2)^2 - 27 = 0$$

Now the expression is in the form of $(x+p)^2+q$ *where* $p = -2$ *and* $q = -27$

6. **Solve** the equation.

$$3(x-2)^2 - 27 = 0$$

Rearrange and solve for x*:*

$$(x-2)^2 = \frac{27}{3}$$
$$(x-2)^2 = 9$$
$$x - 2 = \sqrt{9}$$
$$x - 2 = \pm 3$$
$$x = \pm 3 + 2$$

$$x = 3 + 2 = 5 \qquad \text{or} \qquad x = -3 + 2 = -1$$

Hence, the possible solutions for x are $\boldsymbol{x = 5}$ and $\boldsymbol{x = -1}$.

Solving quadratic equations using the quadratic formula (Higher only)

The quadratic formula can also be used to solve quadratic equations. The quadratic formula is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

And gives solutions to quadratic equations in the form

$$ax^2 + bx + c = 0.$$

Example: Solve $3x^2 - 4x = 11$, giving the answers to 3 significant figures.

1. **Rearrange** the quadratic to be in the general form $ax^2 + bx + c = 0$ and **determine the values** of a, b and c.

$$3x^2 - 4x = 11$$
$$3x^2 - 4x - 11 = 0$$

$$a = 3, \qquad b = -4, \qquad c = -11$$

2. **Substitute** the value of a, b and c into the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(3)(-11)}}{2(3)}$$

$$x = \frac{4 \pm \sqrt{148}}{6}$$

$$x = \frac{4+\sqrt{148}}{6} \qquad \text{or} \qquad x = \frac{4-\sqrt{148}}{6}$$

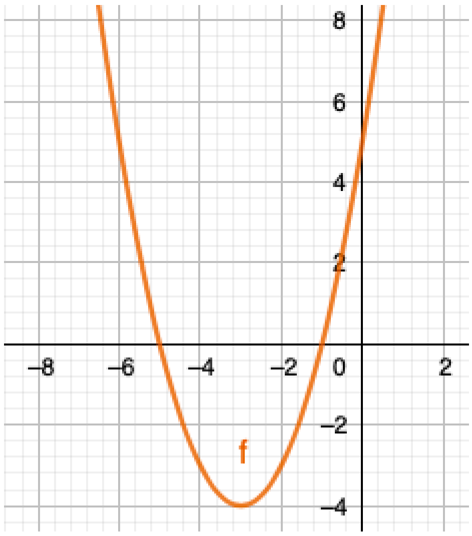
$x = 2.69$ (3.s.f) $\qquad$ $x = -1.36$ (3.s.f)

Hence, the possible solutions for x are $\boldsymbol{x = 2.69}$ and $\boldsymbol{x = -1.36}$.

Finding approximate solutions using a graph

We can find the x −intercept of a quadratic function by setting the equation of the graph equal to 0.

Example: Given the equation $y = x^2 + 6x + 5$, find both of the x −coordinates of the x −intercepts. Use the graph to check your answer.



1. The equation of the graph is $y = x^2 + 6x + 5$.

2. For x −intercepts, $y = 0$.

   Setting $y = 0$ gives us a quadratic equation $x^2 + 6x + 5 = 0$.

3. Factorise the equation to solve the possible values of x.

   $$x^2 + 6x + 5 = 0$$

   The factor pair of 5 which gives a sum of 6 is 5 and 1.

   Hence, the quadratic equation can be expressed in the form of $(x + 1)(x + 5) = 0$.
   That gives us x −coordinates of $x = -1$ and $x = -5$.

4. We can check if these answers are correct from the given graph. With reference to this graph, the line of the graph intercepts the x-axis at $x = -1$ and $x = -5$. Hence, the answer found is indeed correct.

Solving Quadratic Equations – Practice Questions

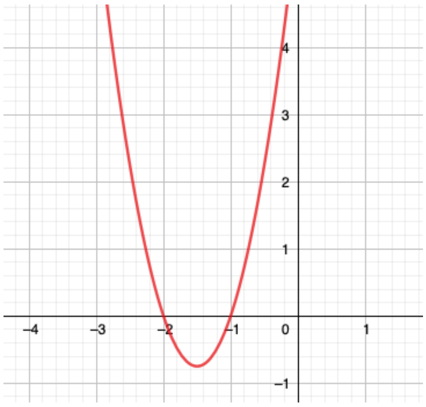
1. Solve the following quadratic equation by factorisation:

$$t^2 - 6t - 27 = 0$$

2. Solve the following quadratic equation by factorisation:

$$x\,(3x + 18) = 120$$

3. Given the graph $y - 6 = 3x^2 + 9x$, find both coordinates of the x –intercepts.



4. Solve the following quadratic equation by factorisation:

$$4x(x - 5) = -25$$

5. (Higher only) Using completing the square method, express the following quadratic equation in the form of $(x + p)^2 + q$. Then, find the exact value for x:

$$x^2 = -5(x + 1)$$

6. (Higher only) Using completing the square method, express the following quadratic equation in the form of $(x + p)^2 + q$. Then, find the exact value for x:

$$-3x\,(x + 1) + 5 = 0$$

7. (Higher only) Solve the following quadratic equation by using the quadratic formula. Give your answer for x to 2 decimal places:

$$(x + 2)^2 = x + 16$$

Worked solutions for the practice questions can be found amongst the worked solutions for the corresponding worksheet file.